Portfolio models - Podgorica

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Outline

Holding period return

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Suppose you invest in a stock-index fund over the next period (e.g. 1 year).
The current price is 100$ per share.
At the end of the period you receive a dividend of 5$; the market price of the stock has increased up to 120$ per share.

The **holding period return** (HPR) is defined as

$$\text{HPR} = \frac{\text{Ending price - Beginning price + Cash dividend}}{\text{Beginning price}} \tag{1}$$

In our example

$$\text{HPR} = \frac{120\$ - 100\$ + 5\$}{100\$} = \frac{25\$}{100\$} = 0.25$$

The HPR for the stck-index fund is therefore 0.25 or 25%.

Holding period return

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The HPR

- assumes that dividends are payed at the end of the holding period (it ignores reinvestments)
- is the sum of *dividend yield* and *capital gain*:

$$\underbrace{\frac{\text{Cash dividend}}{\text{Beginning price}}}_{\text{dividend yield}} + \underbrace{\frac{\text{Ending price - Beginning price}}{\text{Beginning price}}}_{\text{capital gain}}$$

Problem: when you consider an investment at time t, you do not know the future (at time $t+1$) market price and the dividend payed by the stock-index fund. The HPR is not known *with certainty*.

Modeling real world: uncertainty

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The way to model uncertainty about future events is to introduce a *probability structure*

(Discrete state space)

- a set of (possibly infinite) *states of the world*

$$\Omega = \{s_1, s_2, \dots, \}$$

- for each s_i, a measure of *how much likely* is that the world will be in state s_i:

$$\begin{aligned} p_1 &= \text{probability that the actual state will be state 1} = P(s_1) \\ p_2 &= \text{probability that the actual state will be state 2} = P(s_2) \\ &\dots \end{aligned}$$

Probabilities are expressed in numbers $p_i \in [0, 1]$ - e.g. $p_3 = 0.4$ - or in percentages - $p_3 = 40\%$ -, and

$$\sum_{i=1} p_i = 1$$

that is the state space completely describes reality.

Modeling real world: uncertainty

The value of an investment (e.g. the value of a stock) at a future random time is not know in advance with certainty. It can be modeled as a **random variable**, i.e. a variable that may take a different value in each state of the world:

$$\text{random variable V:} \begin{cases} V = v_i \text{ if state is } s_1 & \longrightarrow & \text{proba } p_1 \\ V = v_2 \text{ if state is } s_2 & \longrightarrow & \text{proba } p_2 \\ \quad\quad\quad\quad\quad\quad \ldots & & \end{cases}$$

The **probability distribution** completely describes the random variable V.

Huge amount of information! How do we take any investment decision?

A way to summarize important information about a r.v. is to evaluate the **moments** of the r.v.:

$$\text{E(r) (Expected value of r)} = \sum_s P(s) r_s$$

$$\text{moment of order n of r} = \sum_s P(s) \left[r_s - E(r)\right]^n$$

Expected returns and standard deviation

Consider the same 100$ investment, and the following simple situation: 3 possible scenarios

Scenario	*with probability*	*Ending price*	*Dividend*	*HPR*
Normal growth	0.5	110$	4$	14%
Boom	0.25	130$	5$	35%
Recession	0.25	80$	3.50$	-16.5%

The HPR has become a random variable, whose probability distribution is shown in the last column.

Let's determine the *moments* of return distribution.

- **Expected value of returns** or simply **Expected returns**

$$E(r) = \sum_s P(s) r_s \qquad (2)$$

The first moment is the weighted average of rates of returns in the possible scenarios. The weights are the scenario probabilities (i.e. the probability distribution) therefore a more likely scenario has a higher influence. It is a measure of position.

In our example

$$\begin{aligned} E(r) &= p_1 \cdot r_1 + p_2 \cdot r_2 + p_3 \cdot r_3 \\ &= 14\% \cdot 0.5 + 35\% \cdot 0.25 + (-16.5\%) \cdot 0.25 \\ &= 11.625\% \end{aligned}$$

What does expected return measure? Compare the returns of our index fund with the returns of an alternative investment:

back

Our Index Fund

$$100\$ \rightarrow \begin{cases} 35\% & \text{if boom} \\ 14\% & \text{normal trend} \\ -16.5\% & \text{if recession} \end{cases}$$

Expected returns $= 11.625\%$

Alternative investment

$$100\$ \rightarrow \begin{cases} 45.5\% & \text{if boom} \\ 14\% & \text{normal trend} \\ -27\% & \text{if recession} \end{cases}$$

Expected returns $= 11.625\%$

The two investments share the same expected returns, but the second is *somehow riskier* (in case of recession you would loose a greater amount of money). Expected returns is just a measure of the average reward; it does not take into account how the values of the distribution are dispersed around the mean.

In order to characterize the risk we use second and higher moments.

The second moment is the

- **Variance of returns**

$$Var(r) = \sum_s P(s)\left[r_s - E(r)\right]^2$$

The variance is defined as the expected value of the square deviations of returns around the mean. It is usually indicated as σ^2.
It is a measure of dispersion.

It can be alternatively calculated as

$$Var(r) = E(r^2) - [E(r)]^2 = \sum_s P(s) r_s^2 - \left(\sum_s P(s) r_s\right)^2$$

The square root of the variance, σ, is the **standard deviation**. The standard deviation of returns is a common measure of an asset *risk*.

In the case of our index-fund

$$\begin{aligned}
\sigma^2 &= p_1(r_1 - E(r))^2 + p_2(r_2 - E(r))^2 + p_3(r_3 - E(r))^2 \\
&= 0.5(0.14 - 0.11625)^2 + 0.25(0.35 - 0.11625)^2 + \\
&\quad +0.25(-0.165 - 0.11625)^2 \\
&= 0.5(0.02375)^2 + 0.25(0.18375)^2 + 0.25(-0.28125)^2 \\
&= 0.5\ 0.000564 + 0.25\ 0.033764 + 0.25\ 0.079102 \\
&= 0.028499
\end{aligned}$$

The standard deviation is

$$\sqrt{\sigma^2} = 0.168815 \sim 16.8\%$$

Higher moments

The third moment is the

- **Skewness**

$$Ske(r) = \sum_s P(s)\left[r_s - E(r)\right]^3$$

The contribution of deviations is *cubic*. As a consequence, skewness is dominated by the tails of the distribution.

The *sign* of the skewness is important:

- positive *Ske* indicates large positive deviations from the mean → heavy right tail
- negative *Ske* indicates large negative deviations from the mean → heavy left tail (bad news!). Negative skews imply that the standard deviation underestimates the actual level of risk.

The forth moment is the

- **Kurtosis**

$$Kurt(r) = \sum_s P(s)\left[r_s - E(r)\right]^4$$

The kurtosis is an additional measure of fat tails.

Higher moments

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- **Higher moments**

All higher moments are dominated by the tails of distribution.

Even moments represent likelihood of ”extreme” values. The higher these moments, the higher the dispersion, the higher the uncertainty (risk) about the returns.

Odd moments measure the symmetry of the distribution. Positive odd moments indicate a ”heavier” right tail (where returns are larger than the mean) and are therefore desirable!

Generally, it is a common assumption that well diversified portfolios generate returns that follow a **Gaussian** (Normal) distribution, if the holding period is not too long.

A Gaussian distribution is completely described by the first two moments, mean and variance (or standard deviation).

How much, if anything, should you invest in our index fund?

First, you must ask how much of an *expected reward* is offered to you for the risk involved in investing money in stocks.
The reward is the difference between the expected HPR on the index stock fund and the risk-free rate (T-bills, bank). This is the **risk premium** on common stocks.

$$\text{Risk premium} = E(r) - R_F$$

In our example, if the risk free rate is 5%, the risk premium associated to our index stock found is

$$\text{Risk premium} = 11.625 - 5 = 6.625\%$$

The **excess return** is the difference between the actual rate of return on a risky asset and the risk-free rate:

$$\text{Excess return} = r - R_F$$

Therefore, the risk premium is the expected value of excess return. The risk related associated to excess return can be measured by the standard deviation of excess return.

So, how much should you invest in our index fund?

A complete answer depends on the degree of your risk aversion. But we will assume that you are *risk averse* in the sense that if the risk premium is zero you would not be willing to invest *any* money in risky stocks.

In theory, there must always be a positive risk premium on stocks to induce risk-averse investors to hold the existing supply of stocks instead of placing all their money in risk-free assets.

Analysis of past returns

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- Forward-looking scenarios: determine a set of relevant scenarios; evaluate the rates od return associated to each scenario, as well as the scenario probability. Then compute the risk premium and the risk (std deviation) of investment.
- Time series: we must estimate from historical data the expected return and the risk of investment.

Time series
Consider a time series of N rates of returns of some portfolio/investment

$$r_1,\ r_2, \ldots, r_N$$

over a period of time. How do we evaluate the portfolio performance? How do we estimate portfolio's future expected returns and risk characteristics?

Arithmetic and Geometric average

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Sample (arithmetic) mean

The estimator of the expected return is the arithmetic average of rates of returns:

$$E(r) \sim \bar{r} \triangleq \frac{1}{N} \sum_{i=j}^{N} r_j$$

where N is the number of periods (observations).

Geometric mean

The geometric mean of rates of return is defined as

$$R_G \sim \sqrt[N]{\prod_{i=j}^{N} (1 + r_j)} - 1$$

It is the fixed return that would compound over the period to the same terminal value as obtained from the sequence of actual returns in the time series. Actually

$$(1 - r_1)(1 + r_2) \cdots (1 + r_N) = (1 + R_G)^N$$

- The sample mean is an estimate of future expected returns - the geometric mean is a measure of actual historical performance of investment;
- The geometric mean is always lower than the arithmetic mean;
- If returns are normally distributed, the difference can be estimated as a function of risk:

$$\text{Geometric mean } = \text{Arithmetic mean } - \frac{1}{2}\sigma^2$$

Variance

We would like to estimate the variance of an investment, as an indicator of risk.

- Risk $\rightarrow$ variance: likelihood of deviations from the expected returns
- Estimated risk $\rightarrow$ sample variance: deviations from the *estimated* expected return (arithmetic average)

The sample variance is obtained by assigning the same probability to every observed sample deviation, and using the sample average instead of expectation:

$$\frac{1}{N}\sum_{i=j}^{N}\left(r_j-\bar{r}\right)^2$$

This estimation of variance is downward biased, i.e.

$$E(\sigma^2)-E\left(\sum_{i=j}^{N}\left(r_j-\bar{r}\right)^2\right)>0$$

intuitively because we are using estimated expected returns $\bar{r}$ instead of the true value $E(r)$). The *unbiased* version of sample variance is

$$Var(r) \quad \sim \quad \bar{\sigma}^2 \;\triangleq\; \frac{1}{N-1}\sum_{i=j}^{N}\left(r_j-\bar{r}\right)^2$$

The unbiased sample **standard deviation** is

$$\bar{\sigma} \;\triangleq\; \sqrt{\frac{1}{N-1}\sum_{i=j}^{N}\left(r_j-\bar{r}\right)^2}$$

The Sharpe ratio

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The measures proposed evaluate an investment from the perspective of its expected (total) returns.
When you invest, you are willing to bear same additional risk in order to gain something more than the risk free rate of a T-bill. Investors price risky assets so that the risk premium will be commensurate with the risk of that expected *excess* return.

Sharpe ratio

$$\text{Sharpe ratio (portfolios)} = \frac{\text{Risk premium}}{\text{StdDev of excess return}} = \frac{r - R_F}{\sqrt{\text{Var}(r - R_F)}}$$

From properties of variance we know that

$$\text{Var}(r - R_F) = \text{Var}(r)$$

therefore the Sharpe ratio can be written as

$$\text{Sharpe ratio} = \frac{r - R_F}{\sigma} \sim \frac{\bar{r} - R_F}{\bar{\sigma}}$$

Modeling real world: uncertainty

More tractable models are obtained introducing a *continuum* of states of the world.

(Continuous state space)

- the set of possible states of the world is $\Omega \subset \mathbb{R}$. E.g., it is the realization of a random experiment whose outcome can be *any* number in a given interval.
$$\Omega = [a, b] \ , \quad \Omega = (0, +\infty)$$
We will consider only the case
$$\Omega = \mathbb{R}$$
- we cannot measure $P(\omega)$ (i.e. $P(\omega) = 0$). Instead, we must focus on *sets*: given a set $A \subset \mathbb{R}$, we model the probability that the outcome ω falls in A: $P(A)$.
We must describe $P(A)$ *for every possible set A*.
- this is made via a *density function*:
$$P(A) = \int_{-\infty}^{+\infty} \mathbb{1}_A f(x)\, dx$$

The density function is such that $f(x) \in [0, 1]$ and $\int_{-\infty}^{+\infty} f(x)\, dx = 1$.

Normality assumption

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Analysts often assume that returns from many investments are *normally distributed*. This assumption makes analysis of returns more tractable for many reasons:

- the Gaussian distribution is completely characterized by two parameters (mean and StdDev) $\rightarrow$ simplified scenario analysis; different from other goods
- the Gaussian distribution is symmetric $\rightarrow$ standard deviation is an adequate measure of risk
- the Gaussian distribution is *stable*: when the assets are normally distributed, and you build up a portfolio from them, than the portfolio is also normally distributed.

If a random variable is Normally distributed $N(\mu, \sigma)$, then

- approx. 68% of observations falls in the interval $[\mu - \sigma, \mu + \sigma]$;
- approx. 95% of observations falls in the interval $[\mu - 2\sigma, \mu + 2\sigma]$;
- approx. 99% of observations falls in the interval $[\mu - 3\sigma, \mu + 3\sigma]$;

Outline

Risk aversion

Construction of a portfolio of an investor:

(a) selecting the composition of the risky part of portfolio (stocks composition)

(b) deciding the proportion to invest in that risky portfolio versus in a riskless assets.

Step (a): we assume that the construction of the risky portfolio from the universe of available risky assets has already taken place.

Step(b): the decision of how to allocate investment funds between the risk-free asset and that risky portfolio is based on the risk-return trade-off of the portfolio, and the risk attitude of the investor.

Speculation vs gamble

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We have seen that an investment with 0 expected return will be refused by a risk averse investor. Will an investment that has a positive risk premium always be accepted?

- Speculation is "*the assumption of considerable investment risk to obtain commensurate gain*".

Considerable risk: the potential gain is sufficient to overtake the risk involved. An investment having a positive risk premium can be refused if the potential gain does not make up for the risk involved (in the investor's opinion). Commensurate gain: a positive risk premium.

- A gamble is "*to bet on an uncertain outcome*".

The difference is the lack of a commensurate gain. A gamble is assuming risk for no purpose but enjoyment of the risk itself, whereas speculation is undertaken in spite of the risk involved, because of the perception of a favorable risk-return trade off.

$\rightarrow$ Risk aversion and speculation are not inconsistent.

A *fair game* is a risky investment with a risk premium of zero.
It is a gamble; therefore, a risk-averse investor will reject it.

Risk-averse investors are willing to consider only risk-free or speculative prospects. But might not accept an investment that returns a positive risk premium. Why?

Because investors evaluate investment alternatives not only on returns, but also on risk.
Intuitively a risk averse investor ”penalizes” the expected rate of return of a risky portfolio by a certain percentage to *account for the risk involved*.

In our ▸ previous example the risk-return trade-off is trivial to analyze: as the returns are the same, a risk averse investor will chose the less risky investment.
In other terms we can say that the investor will rank the portfolios using the following preference:

$$\text{Our index fund} \succ \text{Alternative investment}$$

A less trivial example

In the real market **returns increase along with risk**.

Suppose that the risk-free rate in the market is 5% and you have to evaluate the following alternatives

Investment	Expected Returns	Risk premium	Risk (σ)
Low risk	7%	2%	5%
Medium risk	10%	5%	10%
High risk	14%	9%	18%

You need a tool to rank this investments: **Utility function**.

A Utility function is a subjective way to assign scores to the investment alternatives in order to rank them. An investor is *identified by his/her utility function*.

Mean-variance utility

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A commonly used utility function is the **mean-variance utility**: the score of the investment I is

$$U(I) = E(r_I) - \frac{1}{2}A\sigma_I^2$$

- positive effect of returns
- penalty for risk
  - the parameter A is the **risk aversion** of the agent. A risk averse agent has $A > 0$. The more risk averse is the agent, the larger is A. Typical market estimated values of A are between 2 and 5.
  - no penalty for the risk free asset.
- *Note*: expected returns must be expressed in decimals! If you want to use percentages:

$$U(I) = E(r_I) - 0.005A\sigma_I^2$$

Evaluating investments

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Let's evaluate the previous alternatives. Consider an agent characterized by $A = 2$. The agent ranks the investments as follows:

Investment	E(r)	Risk σ	Utility
Low risk (L)	7%	5%	$0.07 - \frac{1}{2}\, 2\, 0.05^2 = 0.0675$
Medium risk (M)	10%	10%	$0.1 - \frac{1}{2}\, 2\, 0.1^2 = 0.09$
High risk (H)	14%	18%	$0.14 - \frac{1}{2}\, 2\, 0.18^2 = 0.1076$

Therefore, for this agent

$$\text{Investment H} \succ \text{Investment M} \succ \text{Investment L}$$

The agent will chose Investment H.

Evaluating investments

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The risk-return characteristics of the investments are *objective* features of the investment.
The Utility score associated to each investment is a *subjective* ranking.

Consider another agent characterized by $A = 5$. The agent ranks the investments as follows:

Investment	E(r)	Risk σ	Utility
Low risk (L)	7%	5%	$0.07 - \frac{1}{2}\,5\,0.05^2 = 0.06375$
Medium risk (M)	10%	10%	$0.1 - \frac{1}{2}\,5\,0.1^2 = 0.075$
High risk (H)	14%	18%	$0.14 - \frac{1}{2}\,5\,0.18^2 = 0.059$

Therefore, for this agent

$$\text{Investment M} \succ \text{Investment L} \succ \text{Investment H}$$

The agent will chose Investment M, because he's more risk averse that the previous agent.

Risky and risk free investments

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- So, why a risk avers would never invest in a fair game?
- Even a risky investment with positive risk premium could be refused:

Investment	Expected return	Risk premium	Risk (σ)
Risky asset	7%	2%	10%
T-bill	5%	0%	0%

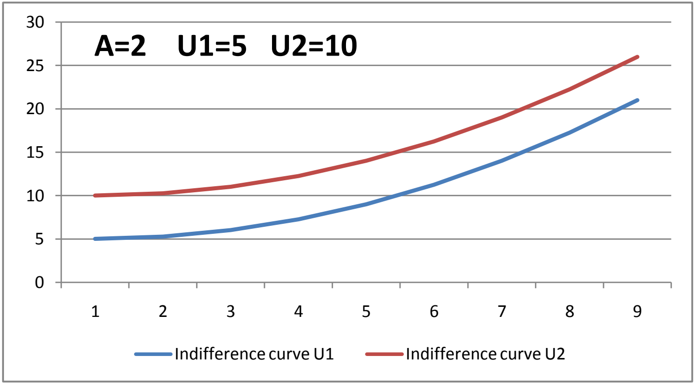
If the risk aversion coefficient is $A = 5$ the scores are

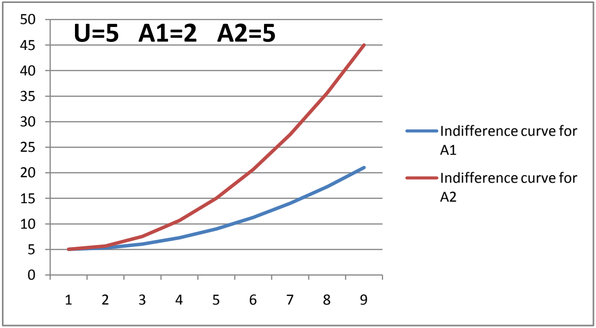
Risky asset $0.07 - \frac{1}{2}\, 5\; 0.1^2 = 0.045$

T-bill $0.05 - \frac{1}{2}\, 5\; 0^2 = 0.05$

indifference curves

Indifference curves are loci such that $U(I)$ =constant





More risk-averse investors have steeper indifference curves; they require a greater risk premium for taking on more risk.

Outline

Portfolios

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A **portfolio** is a collection of financial assets.

When you hold a portfolio, you are interested not only in the individual performances of the asset constituting your portfolio, but also in their "mutual influence": more precisely, you want to measure correlation or covariance.

Consider two assets A and B. The **covariance of returns** is

$$Cov(r_A, r_B) = \sum_s P(s) \left[r_{A,s} - E(r_A)\right] \left[r_{B,s} - E(r_B)\right]$$

In a time series approach, the **sample covariance** is

$$Cov(r_A, r_B) \sim \frac{1}{N} \sum_{i=1}^{N} \left(r_{A,i} - \bar{r}_A\right) \left(r_{B,i} - \bar{r}_B\right)$$

The same information of covariance is carried by the correlation coefficient.

The **correlation coefficient** is defined to be

$$\rho_{AB} = \frac{Cov(r_A, r_B)}{\sigma_A \sigma_B}$$

The correlation coefficient always lies in the interval [-1,1].

The correlation coefficient (and the covariance) is positive if and only if r_A and r_B *lie on the same side of their respective expected returns*. The correlation coefficient is positive if r_A and r_B tend to be simultaneously greater than, or simultaneously less than, $E(r_A)$ and $E(r_B)$ respectively. The correlation coefficient is negative if r_A and r_B tend to lie on opposite sides of the respective expected returns.

Particular cases:

- if $\rho_{A,B} = 1$ the assets are perfectly positively correlated: when r_A increases, r_B increases.
- if $\rho_{A,B} = -1$ the assets are perfectly negatively correlated: when r_A increases, r_B decreases.
- if $\rho_{A,B} = 0$ the assets are uncorrelated. They do not influence each-other.

Two asset case

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Suppose that a portfolio p is composed of stock A and stock B. In particular, denote by w_A the proportion (weight) of wealth invested in the stock A. Therefore, the proportion invested in stock B is $w_B = 1 - w_A$.

The portfolio weights

$$(w_A, w_B)$$

characterize the composition of portfolio.

The **portfolio return** is

$$r_p = w_A r_A + w_B r_B$$

Question. Suppose that you know the expected returns and the variance of each of the two assets A and B. What are portfolio mean and variance, i.e. the mean and variance of the return of the portfolio?

- the **portfolio mean return** is the *weighted average* of returns of stocks:

$$\mu_P = E(r_p) = w_A E(r_A) + w_B E(r_B)$$

- the **portfolio variance** is NOT the weighted average of the two stock variances! Actually:

$$\begin{aligned}
\sigma_P^2 &= Var(r_P) \\
&= w_A^2 Var(r_A) + w_B^2 Var(r_B) + 2 w_A w_B Cov(r_A, r_B) \\
&= w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2\, w_A\, w_B\, \rho_{AB}\, \sigma_A\, \sigma_B
\end{aligned}$$

Exercise. Can you write this expression using just covariances instead of variances?

An interesting example

Example. Consider the following two possible investments:

S&P 500 U.S. Index	9.93%	16.21%
MSCI Emerging Market Index	18.20%	33.11%

The covariance is 0.005. An investor decides to hold a portfolio with 80% invested in the S$P 500 Index, and the remaining 20% in the Emerging Market Index. Evaluate portfolio expected returns and risk.

The portfolio expected return is

$$E(r_p) = w_A E(r_A) + w_B E(r_B) = 0.8\,0.0993 + 0.2\,0.1820 = 0.1158 = 11.58\%$$

It is in between the two asset expected returns.

The portfolio risk is evaluated as σ_p.
Let's start with portfolio variance:

$$\begin{aligned}\sigma_p^2 &= w_A^2 Var(r_A) + w_B^2 Var(r_B) + 2 w_A w_B Cov(r_A, r_B) \\ &= 0.8^2\, 0.1621^2 \;+\; 0.2^2\, 0.3311^2 \;+\; 2\, 0.8\, 0.2\, 0.005 \\ &= 0.02281\end{aligned}$$

The portfolio risk is

$$\sigma_p = \sqrt{0.02281} = 0.15103 = 15.1\%$$

Wait! Let's look at this more closely...

Take an investor who holds the only S&P index, and combine it with a *riskier* asset (Emerging Market Index):

- the expected return increases from 9.93% to 11.58% (intuitive)
- the risk **falls** from 16.21% to 15.10%!

Not only he increases expected returns; but he actually reduces risk! It is the power of **diversification**.

Diversification

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Diversification effect is the risk reduction power of combining assets in a portfolio. How does it work?

Diversification (and risk reduction) is a consequence of *correlation*.

Case A.
Suppose that two assets are perfectly positively correlated, that is $\rho_{AB} = 1$ or $Cov(r_A, r_B) = \sigma_A \sigma_B$.
Then, if we construct a portfolio with weights w_A and w_B, the portfolio variance will be

$$\begin{aligned} \sigma_p^2 &= w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\, w_A\, w_B\, \rho_{AB}\, \sigma_A\, \sigma_B \\ &= w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\, w_A\, w_B\, \sigma_A\, \sigma_B \\ &= (w_A\sigma_A + w_B\sigma_B)^2 \end{aligned}$$

and therefore

$$\sigma_p = w_A\sigma_A + w_B\sigma_B$$

In case of perfect positive correlation, the risk of the portfolio exactly coincides with the weighted average of asset risks. No risk reduction effect.

Case B.
If the assets are not perfectly positively correlated, that $\rho_{AB} < 1$. It means

$$\begin{aligned} \sigma_p^2 &= w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + \rho_{AB}\, 2\, w_A\, w_B\, \sigma_A\, \sigma_B \\ &< w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\, w_A\, w_B\, \sigma_A\, \sigma_B \\ &= (w_A\sigma_A + w_B\sigma_B)^2 \end{aligned}$$

Except for the case of $\rho = 1$ we get $\sigma_p < (w_A\sigma_A + w_B\sigma_B)$: the risk of a diversified portfolio is *lower* than the weighted average of asset risks. Diversification always offers better risk-returns opportunities.

Types of risk:

- the portion of risk than can be reduced (and almost eliminated) through diversification is the *specific* or *idiosyncratic* risk. It is firm specific. It can be ”diversified away”;
- the remaining risk is the *systematic* or *market* risk. It comes from continuous changes in economic conditions. Diversification has no impact in this risk component.

Utility Theory and Portfolio selection

We now apply utility theory to a simple sets of portfolios.

Consider a market in which there exist only two assets

- asset A is a riskless asset (T-bill) whose risk-free rate is R_F
- asset B a risky asset (a stock). Its expected return $E(r)$ must be greater that R_F.

We can build as many portfolios as we like with these 2 assets: it is sufficient to vary the weights w_A and w_B.
For each of these portfolios, the expected value is

$$E(r_p) = w_A\, R_F + w_B\, E(r)$$

while the variance and risk are

$$\begin{aligned}
\sigma_p^2 &= w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\, w_A\, w_B Cov(A, B) \\
&= w_A^2\, 0 + w_B^2\sigma_B^2 + 2\, w_A\, w_B\, 0 \\
&= w_B^2\sigma_B^2 \\
\sigma_p &= w_B\sigma_B
\end{aligned}$$

Now notice that $w_B = 1 - w_A$, therefore we can write

$$\sigma_p = (1 - w_A)\sigma_B$$

and

$$w_A = 1 - \frac{\sigma_p}{\sigma_B} \qquad w_B = \frac{\sigma_p}{\sigma_B}$$

Substitute this expression of w_A in the equation of $E(r)$ and we get

$$E(r_p) = \left(1 - \frac{\sigma_p}{\sigma_B}\right) R_F + \left(\frac{\sigma_p}{\sigma_B}\right) E(r_B)$$

or

$$E(r_p) = R_F + \frac{E(r_B) - R_F}{\sigma_B} \sigma_P$$

$\rightarrow$ this relation is called **capital allocation line**. It is a line in the plane $\sigma_p, E(r_p)$.

Capital allocation line

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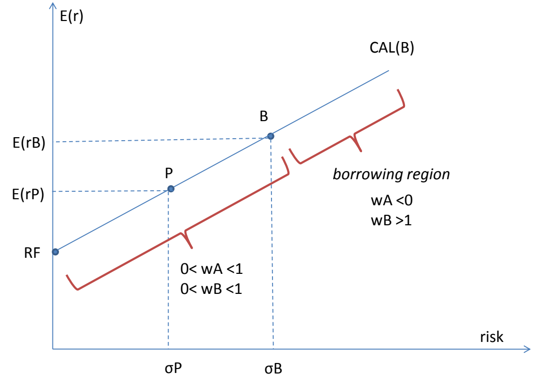

Figure: The Capital Allocation Line CAL(B) describes the set of feasible combinations between the risk-free asset A and a risky investment B

Capital allocation line

- it is the set of all feasible portfolios obtained by combination of a risk-free and a risky investment
- we can move towards higher returns by borrowing money, that is $w_A < 0$ (see next slide)
- points under the capital allocation line might be attainable, but not preferred by any investor
- points above the capital line are desirable but not attainable

Leverage

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If you borrow money:

initial budget	100000$
borrowing	50000$

and you invest all in the risky asset B, then the weight w_B of the risky asset is

$$w_B = \frac{100000 + 50000}{100000} = 1.5$$

and therefore

$$w_A = -0.5$$

When you borrow money you are able to invest positions with higher returns and higher risk:

$$\sigma_P = w_B \sigma_B = 1.5\sigma_B$$

Utility maximization

Once the CAL has been calculated, an investor tries to maximize his/her utility by choosing the risky asset w_B.

$$\begin{aligned}
\max_{w_B} U &= \max_{w_B} \; E(r_P) - \frac{1}{2} A\sigma_P^2 \\
&= \max_{w_B} \; R_F + \frac{E(r_B) - R_F}{\sigma_B}\sigma_P - \frac{1}{2} A\sigma_P^2 \\
&= \max_{w_B} \; R_F + \frac{E(r_B) - R_F}{\sigma_B} w_B \sigma_B - \frac{1}{2} A\, w_B^2 \sigma_B^2 \\
&= \max_{w_B} \; R_F + (E(r_B) - R_F) w_B - \frac{1}{2} A\, w_B^2 \sigma_B^2
\end{aligned}$$

by remembering that $\sigma_p = w_B \sigma_B$.

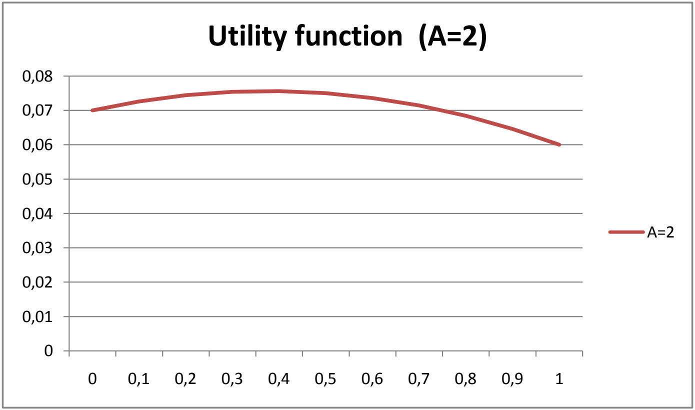

Figure: Case $R_F = 7\%$, risky asset expected return $E(r) = 10\%$ and risk $\sigma = 20\%$.

The maximization is solved by taking the derivative wrt the variable w_B and setting it to zero. We get:

$$w_B^* = \frac{E(r_B) - R_F}{A\sigma_B^2}$$

The optimal position in the risky asset is

- directly proportional to the risk premium
- inversely proportional to the risk of asset, and to the investor's risk aversion.

Graphically this is the tangency point between the CAL and the indifference curves of the investor.

Important remark. These considerations about CAL and utility maximization hold true also if B is a risky portfolio (capital allocation step).

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CAL and utility

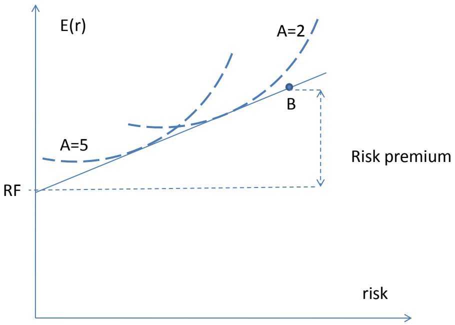

Figure: CAL and optimal investment decisions for different investors

Portfolio of two risky assets

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Now we focus on *risky assets* only and we temporarily rule out the risk-free asset from discussion. We are interested in analyzing how a portfolio of all risky assets works.

As we have seen from previous examples, combining two risky asset in a portfolio may provide risk reduction through diversification.

- if correlation is perfectly positive, there is no risk reduction
- if correlation coefficient is lower than 1, we gain risk reduction. The degree of risk reduction depends on how low is the correlation coefficient.

Perfect negative correlation

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Two stocks A and B have the same return of 10%, and the same risk of 20%. Consider a portfolio made of half the stock A and half the stock B. Calculate portfolio risk and return.

Portfolio return: weighted average of the returns

$$E(r_P) = 0.5\ E(r_A) + 0.5\ E(r_B) = 0.5\ 10\% + 0.5\ 10\% = 10\%$$

Portfolio risk

$$\begin{aligned} \sigma_p^2 &= 0.5^2\ 0.2^2 + 0.5^2\ 0.2^2 + 2\ 0.5\ 0.5\ 0.2\ 0.2(-1) \\ &= 2\ (0.5^2\ 0.2^2) - 2\ (0.5^2\ 0.2^2) \\ &= 0 \end{aligned}$$

When the correlation is perfectly negative, the portfolio can be made **risk free**.

Portfolio of two risky assets

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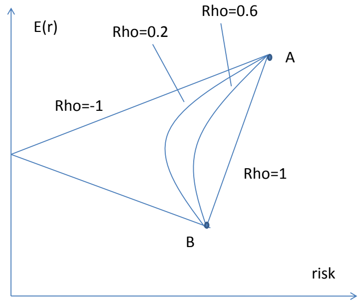

Figure: The combination of two risky assets in a portfolio results in a straight line or a curvilinear combination, depending on the correlation.

Portfolio of many assets

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In real markets, portfolio are made of more than 2 assets. For example, consider the case of a portfolio made of N assets, each of them with weight w_i. The return of such a portfolio is

$$r_p = \sum_{i=1}^{N} w_i \; r_i$$

How do we calculate expected return and risk?

Expected return: it is the weighted average of the expected returns of assets

$$E(r_p) = \sum_{i=1}^{N} w_i \; E(r_i)$$

Variance of returns: it is given by the formula

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \, \sigma_i^2 + \sum_{i,j=1,\; i \neq j}^{N} w_i \, w_j \, Cov(r_i, r_j)$$

Exercise. Do the explicit calculation in the case of 3 assets.

Matrix notations

Matrices help to simplify notations.

Portfolio of N component stocks denoted by $1, 2, \ldots, N$.

Denote

- the proportion vector by

$$\Gamma = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix}; \tag{3}$$

- the vector of asset mean returns by

$$E(r) = \begin{pmatrix} E(r_1) \\ E(r_2) \\ \vdots \\ E(r_N) \end{pmatrix}; \tag{4}$$

- the variance-covariance matrix by

$$S = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2N} \\ \vdots & \vdots & & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NN} \end{bmatrix}. \tag{5}$$

where $Var(r_i) = \sigma_{ii}$ and $Cov(r_i, r_j) = \sigma_{ij}$.

We can write the formulas for the calculation of expected returns and risk of a portfolio of N component stocks using matrix notation.

The portfolio mean return can be written as

$$E(r_P) = \Gamma^T E(r) = E(r)^T \Gamma$$

and the portfolio variance is

$$Var(r_P) = \sum_i \sum_j w_i w_j \sigma_{ij} = \Gamma^T S \Gamma.$$

Exercise. Do the explicit calculations in the case of 2 assets.

Importance of covariances

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In order to understand how a portfolio with N asst works, and how we reduce risk, we write the variance σ_p in another form.

Suppose the portfolio is equally weighted ($w_i = \frac{1}{N}$); let assume that the average asset variance is $\bar{\sigma}$, and the average covariance among assets is $\overline{Cov}$. Then the portfolio variance can be written as

$$\sigma_p^2 = \frac{\bar{\sigma}^2}{N} + \frac{N-1}{N}\overline{Cov}$$

As N increases

- the contribution of assets' variance becomes negligible
- the second term converges to the average covariance $\overline{Cov}$

In a well diversified portfolio, the main contribution to total risk is due to the covariances among assets.

Importance of covariances

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Another insight in the behavior of the portfolio: consider the case in which all the N asset have the same variance σ^2 and the same correlation among them. In this case the previous formula becomes

$$\sigma_p = \sqrt{\frac{\sigma^2}{N} + \frac{N-1}{N}\rho\sigma^2}$$

As N increases

- the contribution of the first term becomes negligible
- therefore $\sigma_p \sim \sqrt{\rho}\sigma$

If asset are not correlated ($\rho = 0$) the portfolio have close to zero risk.

Diversification

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There is a number of approaches for diversification:

- diversify with asset classes (see next table)
- diversify with index mutual funds (less costly)
- diversify among countries
- ...

Table. Correlation among US assets and International Stocks (1970-2008).

International stocks	1.00					
US large company stocks	0.66	1.00				
US small company stocks	0.49	0.71	1.00			
US long-term corporate bonds	0.07	0.31	0.13	1.00		
US long-term treasury bonds	-0.04	0.13	-0.05	0.92	1.00	
US T-bills	-0.02	0.15	0.07	0.02	0.90	1.00

Source: 2009 Ibbotson SBBI Classic Yearbook (Table 13-5)

Diversification

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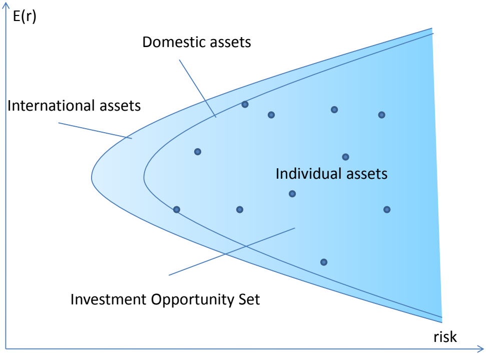
The reason for diversification is simple: by creating a portfolio in which assets do not move together, you can reduce ups and downs in the short period, but benefit from a steady growth in the long term.

We start form one asset and we go on adding other assets in order to gain diversification. The resulting region in the risk-return plane is the Investment Opportunity Set. The limiting curve on the left is an hyperbola. As the number of assets is very high, we suppose that every point in the Investment Opportunity set is attainable by carefully evaluating the assets' proportions to hold.

Adding an asset class means widening the opportunity set.

Diversification and opportunity set



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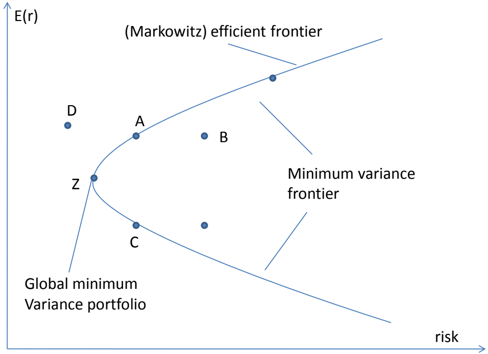
Efficient frontier



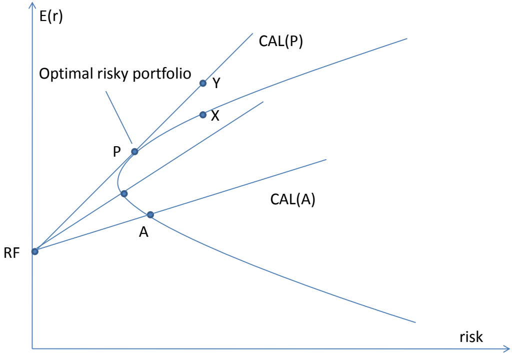
Summary

The main definitions are:

- the **investment opportunity set** is the set of all portfolios obtainable by combinations of one or more investable assets
- the **minimum variance frontier** is the smallest set of (risky asset) portfolios that have a minimum risk for a given expected return
- the **global minimum variance portfolio** is the portfolio with the minimum variance among all portfolios of risky assets. There is NO feasible portfolio of risky assets that has less risk than the global minimum variance portfolio
- the **Markowitz efficient frontier** is the portion of minimum variance frontier that lies above and to the right of the global minimum variance portfolio. It contains *all the portfolios of risky assets that a rational, risk averse investor will choose*

  Important information: the slope of the efficient frontier decreases for higher return investments.

Risk-free asset and optimal efficient portfolio



Back

Two-fund separation theorem

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The **Two-fund separation theorem** states that all investors, regardless of taste, risk preferences and initial wealth will hold a combination of two portfolios: a risk-free asset and an optimal portfolio of risky assets.

The investors' investment problem is divided into two steps:

a the investment decision

b the financing decision

In step *a* the investor identifies the *optimal risky portfolio* among numerous risky portfolios. This is done without any use of utility theory or agent's preferences. It's based just on portfolio returns, risks and correlations.

Once the optimal risky portfolio P is identified, *all* optimal portfolios (i.e. optimal portfolios for any type of agent) must be on the Capital Allocation Line of P (CAL(P)). The optimal portfolio for each investor in determined in step *b* using individual risk preferences (utility).

Example

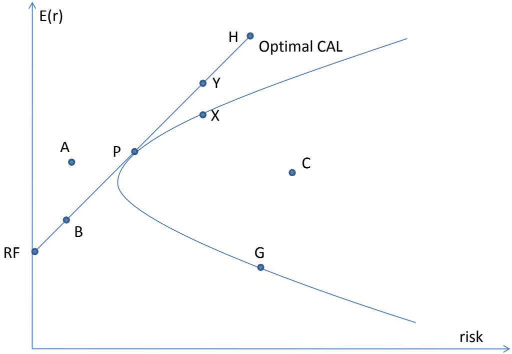
Consider the situation presented in the figure and answer to the following questions.

Among the depicted portfolios

- which ones are *not* achievable?
- which ones will *not* be chosen by a rational, risk-averse investor?
- which one is more suitable for a risk-neutral investor?

G indicates Gold. It is in the non-efficient part of the minimum variance frontier. Why so many rational investors hold gold as a part of their portfolio?

Example



Back

Top-down analysis

- **Capital allocation decision**: choice of the portfolio proportions to be allocated in the riskless (low return) vs risky (higher return) assets.
- **Asset allocation decision**: choice of the broad asset classes (stocks, bonds, real estate,...)
- **Security selection decision**: choice of the specific securities to be hold in each asset class.

Most investment professionals recognize that the asset-allocation decision is the most important decision. A 1991 analysis, which looked at the 10-year results for 82 large pension plans, found that a plan's asset-allocation policy explained 91.5% of the returns earned.

Number of Securities	Expected Portfolio Variance
1	46.619
2	26.839
4	16.948
6	13.651
8	12.003
10	11.014
12	10.354
14	9.883
16	9.530
18	9.256
20	9.036
25	8.640
30	8.376
35	8.188
40	8.047
45	7.937
50	7.849
75	7.585

(Table continues on next slide)

Table 4-8 Effect of Diversification

Elton, Gruber, Brown, and Goetzman: Modern Portfolio Theory and Investment Analysis, Sixth Edition © John Wiley & Sons, Inc.

Number of Securities	Expected Portfolio Variance
100	7.453
125	7.374
150	7.321
175	7.284
200	7.255
250	7.216
300	7.190
350	7.171
400	7.157
450	7.146
500	7.137
600	7.124
700	7.114
800	7.107
900	7.102
1000	7.097
Infinity	7.058

Table 4-8 (continued)

United States	73
U.K.	65.5
France	67.3
Germany	56.2
Italy	60.0
Belgium	80.0
Switzerland	56.0
Netherlands	76.1
International stocks	89.3

Table 4-9 Percentage of the Risk on an Individual Security that Can Be Eliminated by Holding a Random Portfolio of Stocks within Selected National Markets and among National Markets [13]

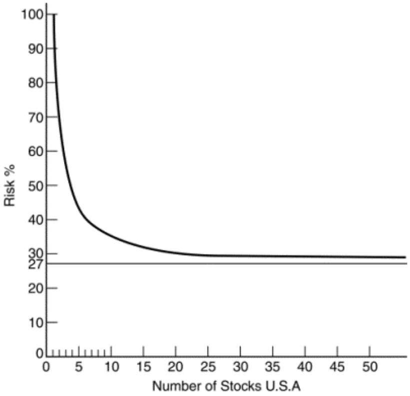


FIGURE 4-2 The effect of number of securities on risk of the portfolio in the United States[13].

Elton, Gruber, Brown, and Goetzman: Modern Portfolio Theory and Investment Analysis, Sixth Edition © John Wiley & Sons, Inc.

Outline

No short sales, no landing/borrowing

Two securities/portfolios case:

- the shape of the curve representing the combination on the mean-std dev plane depends on the correlation among assets (ρ)
- all other attributes held constant, the lower the correlation (ρ close to -1) the higher the payoff from diversification
- the combination has always less risk than that found on a straight line connecting the assets/portfolios in the mean-std dev plane
- a combination might have a risk lower than both single assets. This is NOT ALWAYS the case: it depends on the risk attributes of assets
- the combination produces always a concave curve *above* the minimum variance portfolio, and a convex curve *below* the minimum variance portfolio.

From two to N assets: the efficient frontier

In the case of N assets the possible combinations cover a (bi-dimensional) region of the return-risk plane. Still, we are working under the following assumptions:

- an investor prefers more return to less
- an investor prefers less risk to more

therefore, if we are able to find a set of portfolios that offer

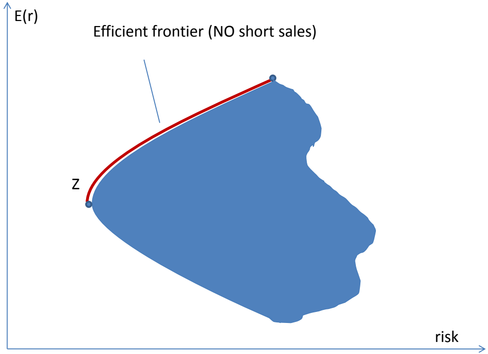
- the highest return for given risk, or
- the lowest risk for a given return

then all other possible portfolios can be ignored.

This efficient set consists of all the envelope curve of portfolios that lie between the global minimum variance portfolio and the maximum return portfolio. It is called **efficient frontier** or **envelope**.

The shape of the efficient frontier must be a concave function

Efficient frontier (no short sales)



Short selling

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Short sale consists in selling a security without owning it.

(A simplified version of) short selling can be described as follows.

Suppose the stock ABC is selling for 100$ per share and pays a annual dividend of 3$ per share. Suppose that you perform some analysis and find put that the stock is likely to sell at 95$ at the end of the year. The (expected) return

$$\frac{95\$ - 100\$ + 3\$}{100\$} = -2\%$$

is negative.

You would not buy the stock! If possible, you would even like to hold a *negative* amount of the stock in your portfolio. How is it possible?

Suppose that your friend Mark owns the stock, and he has different expectations. In particular he is willing to go on holding ABC.
Then you can

- *borrow* the stock from Mark under the promise he will not be worse off
- sell the stock for the current price, receiving 100$
- when ABC pays the 3$ dividend (to the owner of the stock, who is neither you nor Mark) you take 3$ from your pocket and give them to Mark. You must do this because you promised him not to be worse off lending you the ABC stock.
- at the end of the year, you purchase the stock back in the market, and you return it to Mark

If you were right, the purchase price at the end of the year is 95$ and your cash flow is

$$+100\$ - 95\$ - 3\$ = 2\$$$

and you realize a gain.

In reality, the role of Mark is played by a broker. He usually would require an additional compensation (money put as security) for the loan.

Short sales allowed

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Short selling makes sense when the return is negative. But it can make sense even when the return is positive.

	Expected return	Risk (σ)
Stock A	10%	15%
Stock B	5%	10%

A combination P of A and B with weights w_A and w_B produces an expected return of

$$E(r_P) = w_A\ 10\% + w_B\ 5\%$$

Without short sale, the maximum return attainable is 10% ($w_A = 1$, $w_B = 0$).

By short selling the lower return asset B, we can get money to further increase our investment in the higher return asset A: for example

$$w_B = -1 \quad w_A = 2 \quad \rightarrow \quad E(r_P) = 2\ 10\% - 1\ 5\% = 15\%$$

Short sales allowed

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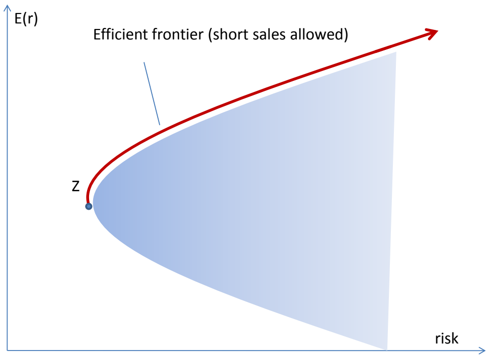
Of course, looking for higher returns means, at the same time, bearing more risk.

Short selling exists in stock markets (eg. New York Stock Exchange)

Many institutional investors do NOT short sell (regulatory or self-imposed constraints).
Anyway, incorporating short sales in our analysis is a minor issue.

Efficient frontier, short sales allowed



Short sales, no lending and borrowing

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Problem: find the portfolio (Markowitz) frontier in the case of N risky assets, no risk-free bond; i.e.
for any target expected return $\bar{R}$, find the portfolio P that realizes this expected return ($E(r_P) = \bar{R}$) and has the lowest variance.

You are given (by a financial analyst) estimates of expected returns (vector $E(r)$) and variance-covariance structure (matrix S) among assets of interest.

Lending and borrowing at riskless rate

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We consider

- lending at riskless rate $R_F \rightarrow$ to invest in a riskless asset (T-bill) of return rate R_F
- borrowing at riskless rate $R_F \rightarrow$ to short sell a riskless asset (T-bill) of return rate R_F

The shape of the combination of a risk-free asset and a risky asset (or portfolio) has been described here

By the *separation theorem*, every rational risk-averse investor will ignore every risky portfolio but one: the tangency portfolio, i.e. the portfolio G such that CAL(G) is tangent to the efficient frontier of risky assets. This portfolio G is the same for each investor, regardless risk-aversion individual characteristics.
All investors will invest in a combination of the riskless asset (R_F) and this optimal portfolio G.

Efficient frontier with riskless landing and borrowing

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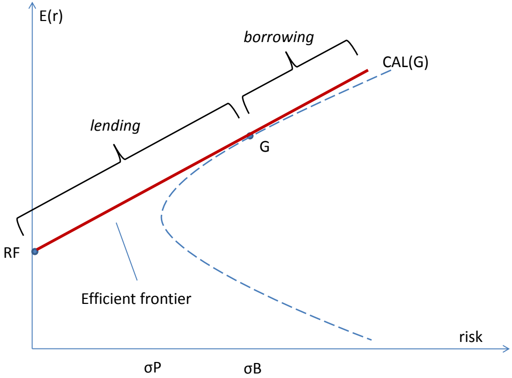

Tangency efficient portfolio

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In this case the efficient frontier coincides with CAL(G). In order to draw it we need only two point in the risk-return plane:

- the risk-free asset
- the tangency portfolio G

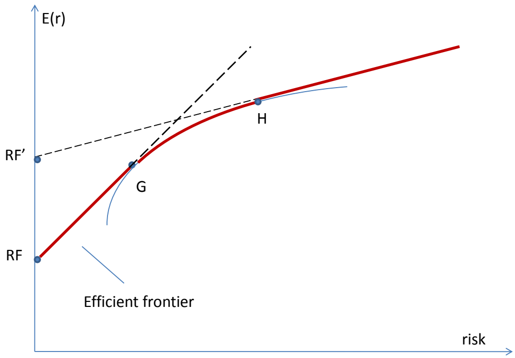
How do you calculate the tangency portfolio G and the efficient frontier?

- G is obtained by the usual procedure by setting $c = R_F$;
- the efficient frontier is the line connecting R_F and G!
  Therefore a portfolio E (efficient) is on CAL(G) iff

$$R_E = R_F + \frac{E(r_G) - R_F}{\sigma_G}\sigma_E$$

We anticipate now that the portfolio G is deeply linked to the so-called Market portfolio.

Different lending and borrowing rates



Efficient frontier under short-sales constraints

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When there is no landing/norrowing (no risk-free asset), the only way to borrow money take a short position in one/more assets.

If short sales are prohibited, than the portfolio optimization problem is more complicated.
The efficient frontier is composed by all solutions to the problem

$$\min_{\Gamma} \Gamma^T S \Gamma$$

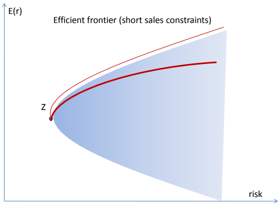
subject to

$$\begin{cases} \sum_{i=1}^{N} w_i = 1 & \text{investor fully invested} \\ \sum_{i=1}^{N} w_i E(r_i) = \bar{R} & \text{target mean return} \\ w_i \geq 0 \forall i & \text{No short sales} \end{cases}$$

Efficient frontier under short-sales constraints

In this case the situation is more complicated

- the optimization problem has additional constraints
- Black's property does *not* hold true: we cannot recover the frontier by combining two frontier portfolios $\rightarrow$ solve the problem for a large number of target returns
- the frontier will in general lie inside the region delimited by the frontier of short-sales-allowed case.



Outline

Input estimation

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The success of a portfolio selection rule strongly depends on the *quality of the input*:

- estimation of expected returns
- estimation of risk and correlation structure among assets

Example. If your security analyst can analyze approx. 50 stocks, it means that your portfolio selection models has the following input:

N = 50	estimates of expected returns
N = 50	estimates of standard deviations
N(N-1)/2 = 1225	estimates of covariances/correlation coefficients

Total: 2325 estimates.

Doubling N to 100 will nearly quadruple the number of estimates; if N=3000 (about NYSE) we need more than 4.5 million estimates!

Input estimation

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Another problem is *consistency* of the estimated correlation structure: error in estimation of the variance-covariance structure can lead to nonsensical results. While *true* variance-covariance matrix is always (obviously) consistent, consistency of *estimated* var-covar matrix must be checked.

The huge amount of data is an actual problem in this sense.

We need to simplify the way we describe the *sources of risk* for securities. Reasons for such a simplification:

- reduction of the set of estimates for risk/returns parameters
- positive covariances among security returns that arise from common economic forces: business cycles, interest rates changes, cost of natural resources, . . .

Decomposition of risk sources

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Idea: decompose uncertainty into *firm-specific* and *system-wide*.

Consider N assets. We can think at the return of each asset as composed by two ingredients:

$$r_i = E(r_i) + \text{unanticipated returns}$$

How do we model the "surprise" term?

- uncertainty about the particular firm: firm specific term e_i
- uncertainty about the economy as a whole. In the simplest model, we can think that this macroeconomic risk can be captured/described by a *single* factor: m.
- the firms are not equally sensitive to the macroeconomic risk m: some react to shocks more than others. We assign each firm a sensitivity factor to macroeconomic conditions: β_i
- coherently with the hypothesis of joint normality of returns, we assume a *linear* relation between the macroeconomic factor and the returns.

The Single Factor model

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For each firm i, the *Single Factor model* describes the returns as

$$r_i = E(r_i) + \beta_i m + e_i$$

Firm specific term: e_i, $i = 1, ..., N$.

- it represents the *innovation* of the firm i. It has subscript i because it is specific of firm i (idiosyncratic term)
- it has zero mean and standard deviation σ_{e_i}:

$$E(e_i) = 0 \qquad Var(e_i) = \sigma^2_{e_i}$$

- idiosyncratic: there is no correlation between e_i and e_j, $i \neq j$:

$$Cov(e_i, e_j) = 0 \quad i \neq j$$

Macroeconomic factor

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Macroeconomic risk term: *m*.

- it has no subscript because it is common to all firms; furthermore

$$E(m) = 0 \qquad Var(m) = \sigma_m^2$$

- it is *uncorrelated* with each term e_i, $i = 1, ..., N$ because e_i is independent from shocks to entire economy.
  As a consequence, the variance of returns is

$$\sigma_{r_i}^2 = \beta_i^2 \sigma_m^2 + \sigma_{e_i}^2$$

- the firm specific terms e_i are independent. The correlation among assets is introduced by the term m: all securities will respond to the same macroeconomic news.

$$Cov(r_i, r_j) = Cov(\beta_i m + e_i, \beta_j m + e_j) = \beta_i \beta_j \sigma_m^2$$

  The covariance structure depends only on the market risk.

Total risk and βs

We have stated that

$$\underbrace{\sigma_{r_i}^2}_{\textit{total risk}} = \underbrace{\beta_i^2 \sigma_m^2}_{\textit{systematic risk}} + \underbrace{\sigma_{e_i}^2}_{\textit{idiosyncratic risk}}$$

The "exposure" of firm *i* to systematic risk is determined by β_i (e.g. cyclical firms).

Problem: how do we choose the common factor?

- this variable must be observable, in order to estimate the volatility
- variance of the common factor usually changes relatively slowly through time, as do the variances of individual securities and the covariances among them.

Single Index model

Common approach: the rate of return on a broad index of securities (e.g. S& P 500) is a valid proxy for the common economic factor behavior:

→ **Single Index Model**

Denote the market index by M: the single index model equation is

$$r_i = E(r_i) + \beta_i r_M + e_i$$

The relation is linear: the sensitivity coefficient can be estimated through a

single-variable linear regression: by defining the excess returns

$$R_i = r_i - R_F$$
$$R_M = r_M - R_F$$

we can use historical data to set up the regression equation

$$R_i(t) = \alpha_i + \beta_i R_M(t) + e_i(t)$$

Interpretation

$$R_i(t) = \alpha_i + \beta_i R_M(t) + e_i(t)$$

The rate of return of each security is the sum of

- α_i: stock's excess return if the market is neutral i.e. if the market excess return is zero
- $\beta_i R_M$: component of excess return due to the movements in the overall market. R_M represent the state of economy; β_i is the sensitiveness to macroeconomic shocks.
- e_i: unexpected movements due to events that are relevant only to stock i

A simple example

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Stock	Capitalization	Beta	Risk premium	Risk (σ)
A	3000$	1	10%	40%
B	1940$	0.2	2%	30%
C	1360	1.7	17%	50%

The market index portfolio is

$$M = \frac{3000}{6300}A + \frac{1940}{6300}B + \frac{1360}{6300}C$$

and suppose it has $\sigma_M = 0.25$.

1. What is covariance between returns of stock A and stock B?
2. What is covariance between returns of stock B and the market index?
3. Decompose the risk of stock B into systematic vs firm-specific component.

A simple example

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Solutions:

- The covariance between A and B is

$$\sigma_{AB} = \beta_A \beta_B \sigma_M^2 = 1 \times 0.2 \times 0.25^2 = 0.0125$$

- The covariance between B and M is

$$\sigma_{BM} = \beta_B \sigma_M^2 = 0.2 \times 0.25^2 = 0.0125$$

- The total risk (variance) of B is

$$\sigma_B^2 = 0.3^2 = 0.09$$

The systematic risk is

$$\textit{Systematic risk} = \beta_B^2 \sigma_M^2 = 0.2^2 \times 0.25^2 = 0.0025$$

The idiosyncratic risk is

$$\begin{aligned} \textit{Idiosyncratic risk} &= \textit{Total risk} - \textit{Systematic risk} \\ &= \sigma_B^2 - \beta_B^2 \sigma_M^2 = 0.09 - 0.0025 = 0.0875 \end{aligned}$$

Portfolios

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In the Single Index model the expected returns, variances and covariances of stocks are

$$\begin{aligned} E(r_i) &= \alpha_i + \beta_i E(r_M) \\ \sigma_i^2 &= \beta_i^2 \sigma_M^2 + \sigma_{e_i}^2 \\ \sigma_{ij} &= \beta_i \beta_j \sigma_M^2 \end{aligned}$$

A portfolio of assets has variance:

$$\sigma_P^2 = \sum_i w_i^2 \beta_i \sigma_i^2 + \sum_i \sum_j w_i w_j \beta_i \beta_j \sigma_M^2 + \underbrace{+ \sum_i w_i^2 \sigma_{e_i}^2}_{\sum_i w_i^2 \left[\sigma_i^2 - \beta_i^2 \sigma_M^2\right]}$$

What do we gain

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$\rightarrow$ the needed parameters are

$$\alpha_i, \beta_i, \sigma_i, E(r_M), \sigma_M$$

that is $3N + 2$ estimates.

If $N \sim 150 - 200$ the model needs about $452 - 602$ estimates.

There is no need to estimate all the covariances among assets: it is sufficient to measure how the assets move with the market.

Portfolio beta

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Define the portfolio beta as the weighted average of the assets beta

$$\beta_P = \sum_i w_i \beta_i$$

and the portfolio alpha as

$$\alpha_P = \sum_i w_i \alpha_i$$

Using these definitions, the portfolio expected return is

$$E(r_P) = \alpha_P + \beta_P E(r_M)$$

and portfolio variance is

$$\sigma_P^2 = \beta_P^2 \sigma_M^2 + \sum_i w_i^2 \sigma_{e_i}^2$$

Risk and diversification

The total risk is $\sigma_P^2 = \beta_P^2 \sigma_M^2 + \sum_i w_i^2 \sigma_{e_i}^2$.

Naive diversification ($w_i = 1/N$):

$$\begin{aligned} \sigma_P^2 &= \beta_P^2 \sigma_M^2 + \sum_i w_i^2 \sigma_{e_i}^2 \\ &= \underbrace{\beta_P^2 \sigma_M^2} + \underbrace{\frac{1}{N} \sum_i \frac{\sigma_{e_i}^2}{N}} \end{aligned}$$

Market risk   Average residual risk

The residual risk (result of idiosyncratic risks) can be made smaller and smaller by increasing N, therefore:

$$\sigma_P^2 \to \beta_P^2 \sigma_M^2 \quad \text{as } N \to \infty$$

Risk and diversification

Number of Securities	Residual Risk (Variance) Expressed as a Percent of the Residual Risk Present in a One-Stock Portfolio with σ^2_{ei} a Constant
1	100
2	50
3	33
4	25
5	20
10	10
20	5
100	1
1000	0.1

Table 7-2 Residual Risk and Portfolio Size

Diversifiable risk

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- In practice the residual risk falls rapidly to zero and can be considered as *eliminated* even for moderate size portfolios
- when moving from the total risk of one asset

  $$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma_{e_i}^2$$

  to the total risk of a *well-diversified* portfolio

  $$\sigma_P^2 \sim \beta_P^2 \sigma_M^2$$

  the contribution of idiosyncratic risks disappears:

  e_i: *diversifiable* risk
- βs are a measure of *non diversifiable* risk: a risk that can not reduced through diversification. For this reason β_i is commonly used as a measure of a security's risk.

Estimation of the Single Index model

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The Single Index model provides a method to estimate the correlation structure among assets. The Single Index model parameters of equation

$$R_i(t) = \alpha_i + \beta_i R_M(t) + e_i(t)$$

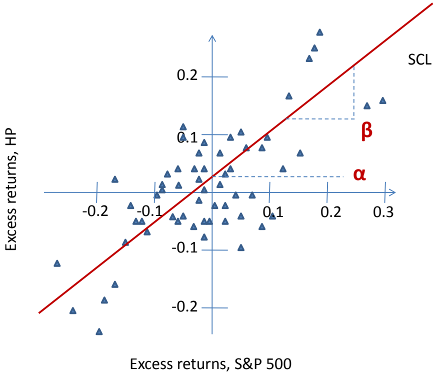
must be efficiently estimated.

The Security Characteristic Line
Consider for example the above equation restated for the Hewlett-Packard stock:

$$R_{HP}(t) = \alpha_{HP} + \beta_{HP} R_{S\&P}(t) + e_{HP}(t)$$

where we used S&P500 as market index. It prescribes a linear dependence of HP excess returns on the excess returns of the S&P index. This relation is known as *Security Characteristic Line*.

Linear regression for HP



Explanatory power of Market Index model

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- empirical tests shows that the correlation between HP and S&P 500 is quite high (~ 0.7): HP tracks S&P changes quite closely
- the R^2 statistic is approx $R^2 \sim 0.5$: a change in S&P 500 excess returns explains about 50% variation in HP excess returns
- the standard error of the regression is the standard deviation of the residual: unexplained portion of HP returns, i.e. portion of returns that is independent from the market index. For HP is about $\sigma_e \sim 26\%$. This is almost equal to the systematic risk ($\beta\sigma_M \sim 27\%$): a common result in individual stock analysis.

Explanatory power of Market Index model

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- intercept α: statistical tests (level of significance) and empirical evidence show that
  - the regressed α is hardly statistically significant (we cannot rely on it in a statistical sense)
  - values of α is not constant over time. We cannot use historical estimates to forecast future values (we cannot rely on it in a economical sense)

  $\rightarrow$ security analysis is not a simple task. Need for adjustment based on forecast models.

- slope β: estimation leads to a value of about $\beta \sim 2$: high sensitivity typical of technology sector (low beta industries: food, tobacco, ...)

- the statistical significance of regressed betas is usually higher; nevertheless, as betas also varies over time, adjustments with forecasts models are necessary.

Portfolio construction

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In practice:

1. Macroeconomic analysis $\rightarrow$ market index analysis (excess returns and risk)
2. Statistical analysis (+ adjustments) $\rightarrow$ sensitivity β and risk specific risk σ_e^2
3. Based solely on these (systematic) information, derive a first estimate of stocks excess returns (absent any firm-specific contribution)
4. Security analysis (security valuation models) $\rightarrow$ estimates of α that captures the firm-specific contribution to excess returns

Many stocks will have similar βs: the are equivalent to the portfolio manager in the sense that the offer the same systematic component of risk premium.

What makes a specific security more attractive is its α: positive and high values of α tell us the security is offering a premium over the premium deriver by simply tracking the market index. This security should be overweighted in an active portfolio strategy (compared to a passive strategy).

Portfolio construction

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Once we have an estimate of αs and βs, we are able to derive the returns and the risk structure (variance-covariance matrix) of the market

$\rightarrow$ we can follow the standard Markowitz procedure in order to find the optimal risky portfolio in the market.

Intuition:

- if we are interested just in diversification, we will just hold the market index i.e. $\alpha_i = 0$ for all i (*passive strategy*)
- performing security analysis and choosing $\alpha_i \neq 0$ can lead to higher returns. At the cost of departure from efficient diversification (assumption of non-necessary additional risk).

In general, the Single Index model over-performs the full-historical-data approach (and β adjustments work even better). The reason is that large part of the observed correlation structure among securities (not considered by the Single Index model) is substantially random noise for forecasts.

Outline

The CAPM: introduction

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- How do the aggregates of investors behave?
- What are the ”fair” returns and prices for assets when market clears?

The Capital Asset Pricing Model gives direct prediction of the relationship between expected returns and risk in equilibrium.

It helps to

- find a ”fair” return for a given risk $\rightarrow$ check for forecast models
- estimate the likely return for an asset which has not be traded yet

Assumptions

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Main idea: investors are *as alike as possible*, with exceptions of initial wealth and risk aversion.

- *perfect competition*: many investors, with small endowed wealth; investor are price-takers
- one holding period; myopic behavior
- investments in publicly traded financial assets (no private enterprises, human capital, ...). Lending and borrowing is allowed at the same, fixed risk-free rate.
- unlimited shot-sales and unlimited lending/borrowing are allowed
- *frictionless market*: no taxes and no transaction costs.
- all investors are rational and have mean-variance preferences $\rightarrow$ they all use Markowitz portfolio selection approach
- *Homogeneity of expectations*: all investor share the same economic view $\rightarrow$ identical scenarios and probability distributions, identical expected returns and variance-covariance structure estimates

Implications

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The goal of CAPM is to start from the simplified word described by above assumptions and to predict the equilibrium that will prevail.

Initial implications: the Market portfolio

Consider the portfolios of all investors, and aggregate them together: lending and borrowing cancel out, and the resulting aggregate portfolio **M** is

- composed by risky asset (stocks)
- the proportion of each stock in M is $\frac{\text{value of the stock}}{\text{aggregate value of stocks}}$

This portfolio is called *market portfolio*. The CAPM predicts that, at equilibrium, all investors solving their portfolio optimization problem will arrive to the same portfolio M.

Market portfolio

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Why?

- investors are homogeneous
- they all use the same approach (Markowitz) applied to the same market, and with the same inputs (estimates)

$\rightarrow$ they all find the same efficient frontier and the same optimal portfolio (the tangency portfolio, i.e. the portfolio whose CAL is tangent to the efficient frontier figure).

This implies that the proportion of each stock in the risky portfolio is the same for every investor: if, foe example, the HP stock is 2% in every optimal risky portfolio, then the total HP stocks is 2% of aggregate portfolio M.

$\rightarrow$ the optimal tangency portfolio from Markowitz *coincides* with (is a share of) the market portfolio M.

The Capital Market Line

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The Capital Market Line (CML) is defined as the CAL of market portfolio M.

We have just seen that, under CAPM, it is the *best attainable CAL*.
All investors' position will be on the CML, differing just by amount invested in the risk-free bond.

For every efficient portfolio it holds

$$E(r_P) = R_F + \frac{E(r_M) - R_F}{\sigma_M}\sigma_P$$

The slope

$$\frac{E(r_M) - R_F}{\sigma_M}$$

is called **market price of risk**: the reward required by investors to the market in order to bear portfolio risk.

Deriving the CAPM

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Intuition: appropriate risk premium for an asset is determined by the its contribution to total risk of the investor's overall portfolio

- *Stock contribution to market variance*
  The variance of market portfolio is

$$
\begin{aligned}
[w_1, ..., w_N]\ S\ [w_1, ..., w_N]^t &= \sum_{i=1}^{N} w_i \left(\sum_{j=1}^{N} w_j Cov(r_i, r_j) \right) \\
&= \sum_{i=1}^{N} w_i\ Cov(r_i, r_M)
\end{aligned}
$$

  Consider one single stock, e.g. HP. The contribution of HP to the total variance is

$$\text{contribution of HP to portfolio variance} = w_{HP}\ Cov(r_{HP}, r_M)$$

Deriving the CAPM

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- *Stock contribution to market risk premium*
  Market risk premium is

$$E(r_M) - R_F \;=\; E\left(\sum_i w_i r_i\right) - R_F \;=\; \sum_i w_i\left(E(r_i) - r_R\right)$$

  The contribution of our HP stock is $w_{HP}\left(E(r_{HP}) - R_F\right)$

- *Marginal price of risk:* for single stock: it is defined as reward-to-risk ratio

$$\frac{\text{contribution to risk premium}}{\text{contribution to variance}}$$

  For HP stock:

$$\frac{w_{HP}\left(E(r_{HP}) - R_F\right)}{w_{HP}\; Cov(r_{HP}, r_M)} = \frac{E(r_{HP}) - R_F}{Cov(r_{HP}, r_M)}$$

Deriving the CAPM

Portfolio models - Podgorica

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Equilibrium: all stocks must offer the same reward-to-risk tradeoff. Therefore, the marginal price of risk of each stock must be *equal to* the market price of risk: e.g. for HP

$$\frac{E(r_{HP}) - R_F}{Cov(r_{HP}, r_M)} = \frac{E(r_M) - R_F}{\sigma_M^2}$$

Rearranging:

$$E(r_{HP}) - R_F = \frac{Cov(r_{HP}, r_M)}{\sigma_M^2} [E(r_M) - R_F]$$

We can define

$$\beta_{HP} = \frac{Cov(r_{HP}, r_M)}{\sigma_M^2}$$

as the contribution of individual stock (HP) as a fraction of total variance of market portfolio.

Return-Beta relationship

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The CAPM expected return-beta relationship states that

$$E(r_{HP}) = R_F + \beta_{HP}\left[E(r_M) - R_F\right]$$

As all investors have the same expectations, they will agree about βs, and also about the required risk premium of each stock.

The relation is true for every combination P of stocks: if we sum up the above relation over all stocks in P, and define the portfolio beta as

$$\beta_P = \sum_i w_i \beta_i$$

then it holds

$$E(r_P) = R_F + \beta_P\left[E(r_M) - R_F\right]$$

Sensitivity and investments

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Beta captures the *systematic risk* component of an investment, describing the sensitivity to market swings. It is the relevant measure of risk at equilibrium.

For the market portfolio

$$\beta_M = \frac{Cov(r_M, r_M)}{\sigma_M^2} = 1$$

Note that β_M can ne seen also as the weighted average beta in the market:

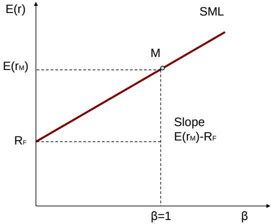
- $\beta > 1$ is an aggressive investment (more volatile than the market, and $E(r_i) > E(r_M)$)
- $\beta < 1$ is a defensive investment (less volatile than the market, and $E(r_i) < E(r_M)$)
- $\beta = 1$ neutral investment

Security Market Line

The risk premium-beta relationship

$$E(r_i) = R_F + \beta_i \left[E(r_M) - R_F\right]$$

is known as Security Market Line.
It states that the risk premium of a stock depends on the contribution of the stock to the total market risk (measure by β) and that the dependence is linear.



Security Market Line

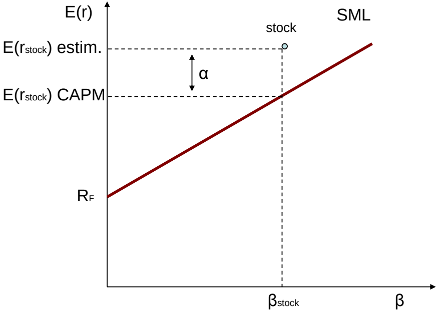
- the Capital Market Line describes the risk premium of an efficient portfolio as a function of its standard deviation
- the SML depicts the risk premium of an individual stock as a function of its risk. If the stock is part of a well diversified portfolio, the right measure of risk is not its standard deviation (because a part of risk, the firm-specific component, has been diversified away) but the contribution of the asset to the total risk, i.e. the asst's beta.

In equilibrium, all fairly priced securities lie on the Security Market Line. In practice, CAPM is used as a benchmark in analysis of under/over-evaluated securities.

under-priced security - excess of return - above the SML

The departure from SML is called the stock's *alpha*.

Security Market Line



Security Market Line

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Security analysis is basically about detecting securities with non-zero alpha.

Portfolio management can take the market portfolio (the *passive strategy*) as a starting point. Then, using security analysis, the manager can overweight those securities that have positive alpha (under-evaluate), and underweight those securities that have negative alpha (over-evaluated).

Capital budgeting decisions: CAPM can be used to estimate the required rate of return of a project (cutoff IRR - hurdle rate) based on its beta.

Test of CAPM

- CAPM is a model that predicts *ex-ante* or **expected** returns. Even in principle it is difficult to test the validity of CAPM
- the Single Index model is, instead, a model that gives a statistical decomposition of *ex-post* or **realized** returns.

We can use the Single Index model to move from realized to expected returns.

We start from the Single Index model: excess returns of stocks follow the relation

$$R_i = \alpha_i + \beta_i R_M + e_i$$

Then we calculate the covariance between the stock and the market index M:

$$Cov(R_i, R_M) = Cov(\alpha_i + \beta_i R_M + e_i, R_M) = \beta_i \sigma_M^2$$

and therefore

$$\beta_i = \frac{Cov(R_i, R_M)}{\sigma_M^2}$$

$\rightarrow$ beta of the Single Index model is *the same* of CAPM, when we replace the (theoretical) market portfolio with a (real, observable) market index.

Test of CAPM

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From Single Index model equation, take the expectation on both sides in order to get a relation for expected returns:

$$E(r_i) - R_F = \alpha_i + \beta_i E(r_M - R_F)$$

The CAPM relation for expected returns is

$$E(r_i) - R_F = \beta_i E(r_M - R_F)$$

$\rightarrow$ Consistency iff alpha are zero. That is, if a stock price is fair, then $\alpha_i = 0$.

More precisely, the CAPM predicts that the expected value of alphas must be zero: it does not mean that in a single period alpha will be identically zero. The realized values of alpha (from a regression) should average to zero.

Statistically significant values far from zero indicate mispriced assets.

In practice, funds with superior performances have positive α; the low number of such funds shows that beating the passive strategy is not easy.

Test of CAPM

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The CAPM fails regression empirical tests. In particular, we can not reject the hypothesis that alphas are different from zero (with an acceptable level of significance).
Data show that low-beta securities have positive alphas, and high-beta securities have negative alphas.

Possible explanations

- failure of market portfolio proxy
- inadequate statistical method
- need for extensions under relaxed hypothesis

What's good of CAPM

- it is simple, as it provides a clear criterion to accept/reject an investment/project
- there is clear evidence that the central conclusion of CAMP, i.e. *the market portfolio is efficient*, may not be far from truth. Money- and tim-costly attempt to beat the market portfolio are performed every day in financial markets, but the number of funds that significantly outperform the passive strategy is still low.

Extensions of CAPM

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- Black: restrictions on riskless borrowing $\rightarrow$ investor optimal portfolio might deviate from market portfolio. This extension allows for negative/positive betas
- Merton:
  - relaxing the myopic hypothesis of one single period $\rightarrow$ *intertemporal CAPM*. It predicts the same expected return-beta relationship as the single-period equation.
  - adding different sources of risk: interest rates changing unpredictably over time (interest rate risk), prices of consumption goods rise (inflation risk), etc.

    Adding K extra-market sources of risk means identifying K hedge portfolios, and leads to a multi-factor SML equation

$$E(R_I) = \beta_{IM} E(R_M) + \sum_{k=1}^{K} \beta_{Ik} E(R_k)$$

Extensions of CAPM

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- Rubinstein, Lucas and Breeden: Consumption based CAPM (CCAPM)

  The investor decides his lifetime plane; she must optimally allocate his current wealth among

  - today's consumption
  - savings and investment that supports future consumption.

  Investment (moving your money from savings to market) in risky assets is a way to increase the expected *consumption growth*. How is the *risk* determined?

  Main idea: additional income (from investment) is valued *more* during difficult economic times (scarce consumption opportunities) then in affluent times (consumption already abundant).
  Therefore an asset is

  riskier $\leftrightarrow$ its returns has positive correlation with consumption growth

  i.e. payoff is higher when consumption is already high, and lower when consumption already low.

Extensions of CAPM

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In this model: higher correlation with consumption growth results in higher risk premia:

$$E(R_I) = \beta_I R_{consumption-mimiking \quad portfolio}$$

- analogous set-up to the standard CAMP
- data about consumption growth are published infrequently, and measured less accurately.
- nevertheless, CCAPM is more successful in explaining realized returns than CAPM